

A JULIA–CARATHÉODORY THEOREM
FOR HYPERBOLICALLY MONOTONE MAPPINGS
IN THE HILBERT BALL

BY

MARK ELIN

*Department of Mathematics, ORT Braude College,
21982 Karmiel, Israel
e-mail: mark.elin@gmail.com*

AND

SIMEON REICH*

*Department of Mathematics,
The Technion — Israel Institute of Technology,
32000 Haifa, Israel
e-mail: sreich@tx.technion.ac.il*

AND

DAVID SHOIKHET

*Department of Mathematics, ORT Braude College,
21982 Karmiel, Israel
e-mail: davs27@netvision.net.il*

ABSTRACT

We establish a Julia–Carathéodory theorem and a boundary Schwarz–Wolff lemma for hyperbolically monotone mappings in the open unit ball of a complex Hilbert space.

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Let \mathbb{B} be the open unit ball of a complex Hilbert space \mathcal{H} with an inner product $\langle \cdot, \cdot \rangle$ and an induced norm $\| \cdot \|$, and let $\rho : \mathbb{B} \times \mathbb{B} \mapsto \mathbb{R}^+$ be the hyperbolic metric on \mathbb{B} ([14], p. 98), i.e.,

$$(1) \quad \rho(x, y) = \tanh^{-1} \sqrt{1 - \sigma(x, y)},$$

where

$$(2) \quad \sigma(x, y) = \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y \rangle|^2}, \quad x, y \in \mathbb{B}.$$

We denote by \mathcal{N}_ρ the class of all those self-mappings $F : \mathbb{B} \mapsto \mathbb{B}$ which are nonexpansive with respect to ρ (ρ -nonexpansive), i.e.,

$$(3) \quad \rho(F(x), F(y)) \leq \rho(x, y).$$

Note that the class \mathcal{N}_ρ properly contains the class $\text{Hol}(\mathbb{B})$ of all holomorphic self-mappings of \mathbb{B} ([13, 14]).

Definition 1: A family $\mathcal{S} = \{F(t)\}_{t \geq 0}$ of self-mappings of \mathbb{B} is said to be a one-parameter continuous semigroup (flow) on \mathbb{B} if

$$(4) \quad F(t + s) = F(t) \circ F(s), \quad t, s \geq 0,$$

and

$$(5) \quad \lim_{t \rightarrow 0^+} F(t) = I,$$

where I is the restriction of the identity mapping of \mathcal{H} to \mathbb{B} and the limit is taken pointwise with respect to the strong topology of \mathcal{H} .

We denote $F(t)x$, the value of $F(t)$ at $x \in \mathbb{B}$, by $F_t(x)$, $t \geq 0$.

Definition 2: A flow $\mathcal{S} = \{F(t)\}_{t \geq 0}$ on \mathbb{B} is said to be generated if for each $x \in \mathbb{B}$, there exists the strong limit

$$(6) \quad f(x) := \lim_{t \rightarrow 0^+} \frac{1}{t}(x - F_t(x)).$$

In this case the mapping $f : \mathbb{B} \mapsto \mathcal{H}$ is called the (infinitesimal) generator of \mathcal{S} .

If f generates a flow of ρ -nonexpansive self-mappings of \mathbb{B} , then we will write $f \in \mathcal{GN}_\rho(\mathbb{B})$.

The following result is established in [20]:

◆ *A semigroup \mathcal{S} of holomorphic self-mappings of \mathbb{B} is differentiable with respect to the parameter $t \geq 0$ (hence, generated by a holomorphic mapping) if and only if it is locally uniformly continuous on \mathbb{B} , i.e., the limit in Definition 1 is uniform on each ρ -ball in \mathbb{B} .*

Moreover, in this case (see [15] and [21]):

◆ *The generator f is holomorphic on \mathbb{B} , and bounded and uniformly continuous on each ρ -ball in \mathbb{B} .*

The set of all holomorphic semigroup generators is denoted by $\mathcal{G}\text{Hol}(\mathbb{B})$.

The classical Julia–Carathéodory theorem and the boundary Schwarz–Wolff lemma play a crucial role in geometric function theory (see, for example, [8] and [25]). In particular, they can be effectively used in the study of the asymptotic behavior of discrete and continuous dynamical systems. In this context these celebrated results may be stated as follows:

◆ *Let F be a holomorphic self-mapping of the open unit disk Δ in the complex plane \mathbb{C} . If for a boundary fixed point $\tau \in \partial\Delta$ ($\lim_{r \rightarrow 1^-} F(r\tau) = \tau$) the angular derivative $\angle F'(\tau) := \angle \lim_{z \rightarrow \tau} F'(z) = k$ exists finitely, then*

$$\frac{|F(z) - \tau|^2}{1 - |F(z)|^2} \leq k \frac{|z - \tau|^2}{1 - |z|^2}.$$

- *If $k \leq 1$ this inequality means that each horocycle internally tangent to the unit circle $\partial\Delta$ at $\tau \in \partial\Delta$ is F -invariant.*
- *This is indeed the case when F has no fixed point in Δ and $\tau \in \partial\Delta$ is its so-called Denjoy–Wolff point, that is, τ is an attractive fixed point for all orbits of F .*

We use the symbol $\angle \lim_{z \rightarrow \tau}$ to denote the limit in each nontangential approach region (see, for example, [18] and [25]).

Sometimes the above statements are grouped together under the name **the Julia–Wolff–Carathéodory theorem**. Higher dimensional analogs can be found, for instance, in [24, 14, 9, 1, 8, 3, 2].

For holomorphic mappings on the open unit disk Δ in the complex plane \mathbb{C} (i.e., for the one dimensional case when $\mathbb{B} = \Delta$), an infinitesimal version of the Julia–Wolff–Carathéodory theorem was given in [11]. Namely, it was shown there that

◆ $f \in \mathcal{G}\text{Hol}(\Delta)$ has no null point in Δ if and only if for some $\tau \in \partial\Delta$, the angular derivative

$$(7) \quad \angle f'(\tau) := \angle \lim_{z \rightarrow \tau} f'(z) = \alpha$$

exists (finitely) with $\text{Re } \alpha \geq 0$.

Moreover, α is, in fact, real and if $\mathcal{S} = \{F(t)\}_{t \geq 0}$ is the semigroup generated by f , then

$$(8) \quad \frac{|F_t(z) - \tau|^2}{1 - |F_t(z)|^2} \leq \exp(-t\alpha) \frac{|z - \tau|^2}{1 - |z|^2}, \quad z \in \Delta, t \geq 0.$$

The point τ is unique and a (globally) attractive sink point of \mathcal{S} .

It is worth mentioning that the original Julia–Carathéodory theorem deals not only with attractive boundary fixed points, but also with repelling fixed points (see [8, 22]), i.e., it deals with not necessarily fixed point free holomorphic self-mappings. In this direction, a generalization of the above theorem has recently been given by M. D. Contreras, S. Díaz-Madrigal and Ch. Pommerenke [7]. They proved the following one-dimensional assertion:

◆ Let $f \in \mathcal{G}\text{Hol}(\Delta)$. For a boundary point $\tau \in \partial\Delta$, the following claims are equivalent:

- (i) the angular limit $\angle f'(\tau) := \angle \lim_{z \rightarrow \tau} \frac{f(z)}{z - \tau} = \beta$ exists finitely;
- (ii) the angular limit $\frac{dF_t(\tau)}{dz} := \angle \lim_{z \rightarrow \tau} \frac{F_t(z) - \tau}{z - \tau} = \exp(-t\beta)$.

Note that condition (ii) is equivalent to inequality (8) for some $\alpha \leq \beta$.

In this paper we will establish these assertions for a general complex Hilbert (not necessarily finite-dimensional) space \mathcal{H} . Moreover, we will show that replacing the angular derivatives by just radial derivatives, we are able to prove infinitesimal versions of the Julia–Carathéodory theorem and the boundary Schwarz–Wolff lemma for the much wider class of generators of semigroups of ρ -nonexpansive self-mappings of the Hilbert ball \mathbb{B} . For the case of the Denjoy–Wolff point, the asymptotic behavior of one-parameter semigroups of ρ -nonexpansive and holomorphic mappings was also studied in [10].

Indeed, the content of the classical Schwarz–Pick lemma is the fact that each holomorphic self-mapping of the open unit disk in the complex plane is nonexpansive with respect to the Poincaré hyperbolic metric ρ defined by (1). Therefore, one can try to use metric fixed point theory to derive results regarding those mappings which are nonexpansive with respect to ρ . At the same time,

we must remember that if a given mapping (or semigroup) is not holomorphic, then the notion of derivative makes no sense.

It turns out, however, that although the study of the asymptotic behavior of a fixed point free semigroup consisting of ρ -nonexpansive mappings is, in general, much more complicated, one can define the real part of the radial derivative of its generator at a boundary fixed point and use it to find invariant ellipsoids, internally tangent at this point to the unit sphere, in the spirit of the Julia–Carathéodory theorem and the boundary Schwarz–Wolff lemma. To understand this phenomenon, we first consider the following example.

Example 1: Consider the continuous function $f : \overline{\Delta} \mapsto \mathbb{C}$ defined by

$$f(z) = \frac{z + \bar{z}}{2} + \chi \frac{z - \bar{z}}{2} - 1,$$

where χ is a real parameter. Elementary calculations show that for all $\chi \geq 1/2$ the following boundary flow invariance condition holds:

$$\operatorname{Re} f(z)\bar{z} \geq 0, \quad z \in \partial\Delta.$$

Therefore it follows from Martin’s theorem [17] (see also [19]) that f generates a semigroup of continuous self-mappings of $\overline{\Delta}$. Indeed, solving the Cauchy problem

$$(9) \quad \begin{cases} \frac{\partial F_t(z)}{\partial t} + f(F_t(z)) = 0, \\ F_0(z) = z, \quad z \in \Delta, \end{cases}$$

we find

$$F_t(z) = 1 - e^{-t} + e^{-t} \frac{z + \bar{z}}{2} + e^{-\chi t} \frac{z - \bar{z}}{2}, \quad z \in \overline{\Delta}.$$

It is clear that $F_t(1) = 1$ for all $t \geq 0$ and $\lim_{t \rightarrow \infty} F_t(z) = 1$ for all $z \in \overline{\Delta}$.

However, no more information on invariant sets of this semigroup can be obtained in this way.

If $\chi = 1$, then for each $t \geq 0$, the function $F_t(z)$, as well as f , are holomorphic. Hence one can apply the result in [11] (see (7)–(8) above) to derive

$$(10) \quad d_1(F_t(z)) := \frac{|1 - F_t(z)|^2}{1 - |F_t(z)|^2} \leq \frac{e^{-t}|1 - z|^2}{1 - |z|^2},$$

because the angular derivative of f at the boundary fixed point $\tau = 1$ exists and equals 1.

But for $\chi > 1$ our generator f , as well as its generated semigroup, are again not holomorphic.

However, fortunately, in this situation ($\chi > 1$) one can show that the semigroup $\mathcal{S} = \{F_t\}_{t \geq 0}$ consists of ρ -nonexpansive self-mappings of the open unit disk Δ with the Poincaré hyperbolic metric ρ defined on it. Indeed,

$$F_t(z) = (1 - e^{-t}) + z(e^{-t}/2 + e^{-\chi t}/2) + \bar{z}(e^{-t}/2 - e^{-\chi t}/2)$$

is a convex combination of holomorphic and anti-holomorphic (hence, ρ -nonexpansive) mappings. So, by Theorem 6.5 on page 75 of [14], F_t is also ρ -nonexpansive.

In addition, we see that although f is not differentiable in the complex sense in Δ , its radial derivative at the boundary null point $\tau = 1$ does exist:

$$\alpha := \lim_{r \rightarrow 1^-} f(r)/(r - 1) = 1.$$

So, the following question arises:

- *Is this fact sufficient to ensure the same invariance condition as (10) for a semigroup of ρ -nonexpansive self-mappings of the open unit disk Δ ?*

In this paper we give an affirmative answer to this question in a much more general situation.

To formulate our results, we need the following notions and notations.

For a fixed $\tau \in \partial\mathbb{B}$, the boundary of \mathbb{B} , and an arbitrary $x \in \mathbb{B}$, we define a non-Euclidean “distance” between x and τ by the formula

$$(11) \quad d_\tau(x) := \frac{|1 - \langle x, \tau \rangle|^2}{1 - \|x\|^2}.$$

The sets

$$(12) \quad E(\tau, s) := \left\{ x \in \mathbb{B} : d_\tau(x) = \frac{|1 - \langle x, \tau \rangle|^2}{1 - \|x\|^2} < s \right\}, \quad s > 0,$$

are ellipsoids internally tangent to the unit sphere $\partial\mathbb{B}$ at τ .

As in [4], it can be shown that the support functional x^* of the smooth convex set $E(\tau, s)$ at $x \in \partial E(\tau, s)$, $x \neq \tau$, normalized by the condition

$$\lim_{x \rightarrow \tau} \langle x - \tau, x^* \rangle = 1,$$

can be expressed by

$$(13) \quad x^* = \frac{1}{1 - \|x\|^2} x - \frac{1}{1 - \langle \tau, x \rangle} \tau.$$

Remark 1: If the generator $f : \mathbb{B} \mapsto \mathcal{H}$ is holomorphic, a result similar to the following Theorem can be found in [12, Theorem 1.1]. However, the application of the Theorem in [5] in Step 2 of the proof of [12, Theorem 1.1] is not justified. We avoid this gap in the present proof of our more general result. In the finite dimensional case, a different, powerful technique has recently been used in [6] to prove several related results (see, in particular, Theorem 4.7 there).

THEOREM: *Let \mathcal{S} be a semigroup of ρ -nonexpansive self-mappings of the Hilbert ball \mathbb{B} , generated by $f : \mathbb{B} \mapsto \mathcal{H}$. Suppose that f is uniformly continuous on each ρ -ball in \mathbb{B} , and $\tau \in \partial\mathbb{B}$ is a null point of f in the sense that $\lim_{r \rightarrow 1^-} f(r\tau) = 0$.*

The following assertions are equivalent:

(I)

$$(14) \quad \limsup_{r \rightarrow 1^-} \operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r - 1} > -\infty;$$

(II) $\alpha := \lim_{r \rightarrow 1^-} \operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r - 1}$ exists finitely;

(III) $\beta := \inf 2 \operatorname{Re} \langle f(x), x^* \rangle > -\infty$;

(IV) there exists a real number γ such that

$$(15) \quad d_\tau(F_t(x)) \leq \exp(-t\gamma) \cdot d_\tau(x), \quad x \in \mathbb{B}.$$

Moreover,

(a) $\alpha = \beta$ and the maximal γ for which (IV) holds is exactly the same β ;

(b) if f is holomorphic and one (hence all) of conditions (I)–(IV) holds, then

$$\lim_{r \rightarrow 1^-} \frac{\langle f(r\tau), \tau \rangle}{r - 1}$$

exists and is actually a real number equal to $\alpha = \beta$.

Combining this result with Theorem 8.3 in [23], we arrive at the following analog of the boundary Schwarz–Wolff lemma.

Corollary: Let $\mathcal{S} = \{F_t\}_{t \geq 0} \subset \mathcal{N}_\rho$ be a semigroup of ρ -nonexpansive self-mappings of \mathbb{B} generated by $f \in \mathcal{GN}_\rho(\mathbb{B})$. Assume that f is null point free (that is, \mathcal{S} has no stationary point in \mathbb{B}). Then there is a unique point $\tau \in \partial\mathbb{B}$ such that

$$d_\tau(F_t(x)) \leq \exp(-t\alpha) \cdot d_\tau(x), \quad x \in \mathbb{B},$$

for some $\alpha \geq 0$, and there is a continuous curve $\{x(r) : 0 \leq r < 1\} \subset \mathbb{B}$ ending at $\tau \in \partial\mathbb{B}$ for which

$$\lim_{r \rightarrow 1^-} f(x(r)) = 0.$$

Conversely, if for some point $\tau \in \partial\mathbb{B}$ the radial limit $\lim_{r \rightarrow 1^-} f(r\tau) = 0$ and the radial limit $\alpha := \lim_{r \rightarrow 1^-} \operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r-1}$ exists and is positive, then for each $t > 0$, the mapping F_t has no fixed point in \mathbb{B} .

To prove our Theorem we need the following additional concepts and facts.

A mapping $f : \mathbb{B} \mapsto \mathcal{H}$ is said to be hyperbolically monotone or ρ -monotone ([21, 16]) if for each pair $x, y \in \mathbb{B}$,

$$(16) \quad \rho(x + rf(x), y + rf(y)) \geq \rho(x, y)$$

for all $r \geq 0$ such that the points $x + rf(x)$ and $y + rf(y)$ belong to \mathbb{B} .

The crucial point in our approach is the following characterization of ρ -monotonicity [21, Theorem 2.1]:

◆ *A mapping $f : \mathbb{B} \mapsto \mathcal{H}$ is ρ -monotone if and only if*

$$(17) \quad \operatorname{Re} \left[\frac{\langle f(x), x \rangle}{1 - \|x\|^2} + \frac{\langle y, f(y) \rangle}{1 - \|y\|^2} \right] \geq \operatorname{Re} \left[\frac{\langle f(x), y \rangle + \langle x, f(y) \rangle}{1 - \langle x, y \rangle} \right]$$

for all points $x, y \in \mathbb{B}$.

Moreover, if $f \in \mathcal{GN}_\rho(\mathbb{B})$ is uniformly continuous on each ρ -ball in \mathbb{B} , then it is ρ -monotone.

So, each holomorphic generator on \mathbb{B} is ρ -monotone.

Proof of Theorem. The implication (II) \Rightarrow (I) is trivial. To prove other implications we denote by ψ the following real-valued function:

$$\psi(t, x) := d_\tau(F_t(x)).$$

By direct calculation we get

$$(18) \quad \left. \frac{\partial \psi(t, x)}{\partial t} \right|_{t=0^+} = -2\psi(0, x) \operatorname{Re} \langle f(x), x^* \rangle,$$

where x^* is defined by (13). Hence, by the semigroup property (4), $\psi(t + s, x) = \psi(s, F_t(x))$ and

$$\begin{aligned} \frac{\partial^+ \psi(t, x)}{\partial t} &:= \lim_{s \rightarrow 0^+} \frac{\psi(t + s, x) - \psi(t, x)}{s} = \lim_{s \rightarrow 0^+} \frac{\psi(s, F_t(x)) - \psi(0, F_t(x))}{s} \\ &= \left. \frac{\partial \psi(s, F_t(x))}{\partial s} \right|_{s=0^+} \\ &= -2\psi(t, x) \operatorname{Re} \langle f(F_t(x)), (F_t(x))^* \rangle \end{aligned}$$

for each $t \geq 0$. Since this right-hand derivative is continuous, it follows (see, for example, [27, p. 239]) that ψ is, in fact, differentiable at each $t \geq 0$ and

$$(19) \quad \frac{\partial \psi(t, x)}{\partial t} = -2\psi(t, x) \operatorname{Re} \langle f(F_t(x)), (F_t(x))^* \rangle$$

for all $t \geq 0$.

STEP 1: (I) \Rightarrow (III). If (I) holds, then there exists a sequence $\{r_n\}$ such that $r_n \nearrow 1$ and

$$\alpha_1 := \limsup_{r \rightarrow 1^-} \operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r - 1} = \lim_{n \rightarrow \infty} \operatorname{Re} \frac{\langle f(r_n\tau), \tau \rangle}{r_n - 1}.$$

Setting $y = r_n\tau$ in (17), we get

$$\operatorname{Re} \left[\frac{\langle f(x), x \rangle}{1 - \|x\|^2} + \frac{\langle r_n\tau, f(r_n\tau) \rangle}{1 - r_n^2} \right] \geq \operatorname{Re} \left[\frac{\langle f(x), r_n\tau \rangle + \langle x, f(r_n\tau) \rangle}{1 - \langle x, r_n\tau \rangle} \right],$$

or, equivalently,

$$\begin{aligned} \operatorname{Re} \left\langle f(x), \frac{x}{1 - \|x\|^2} - \frac{r_n\tau}{1 - r_n \langle \tau, x \rangle} \right\rangle &\geq \\ &\operatorname{Re} \left\langle f(r_n\tau), \frac{x}{1 - r_n \langle x, \tau \rangle} - \frac{r_n\tau}{1 - r_n^2} \right\rangle. \end{aligned}$$

Letting now n tend to ∞ , we see that

$$\operatorname{Re} \langle f(x), x^* \rangle \geq \alpha_1/2,$$

i.e., (III) holds and β is not less than α_1 . By the way, this implies that α_1 is finite.

STEP 2: (III) \Leftrightarrow (IV). Let (IV) hold for some real number γ . Differentiating (15) at $t = 0$, we obtain

$$2 \operatorname{Re} \langle f(x), x^* \rangle \geq \gamma,$$

i.e., (III) holds, and $\beta \geq \gamma$.

Let now (III) hold. Then by (19),

$$\frac{\partial \psi(t, x)}{\partial t} \leq -\psi(t, x)\beta.$$

Integrating this inequality with respect to t , we see that (IV) holds with $\gamma = \beta$.

STEP 3: (III) \Rightarrow (I). Let

$$2 \operatorname{Re} \left\langle f(x), \frac{x}{1 - \|x\|^2} - \frac{\tau}{1 - \langle \tau, x \rangle} \right\rangle \geq \beta.$$

Substituting $x = r\tau$, $0 < r < 1$, we see that

$$2 \operatorname{Re} \left\langle f(r\tau), \frac{r\tau}{1 - r^2} - \frac{\tau}{1 - r} \right\rangle \geq \beta,$$

or

$$\operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r - 1} \cdot \frac{2}{r + 1} \geq \beta.$$

Therefore

$$\liminf_{r \rightarrow 1^-} \operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r - 1} \geq \beta.$$

Hence (I) holds and $\limsup_{r \rightarrow 1^-} \operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r - 1} \geq \beta$.

STEP 4: Just by comparing Step 1 and Step 3, we conclude that (III) implies (II), i.e., if (III) holds, then $\alpha = \lim_{r \rightarrow 1^-} \operatorname{Re} \frac{\langle f(r\tau), \tau \rangle}{r - 1}$ exists and is equal to β .

STEP 5: To end the proof, we have to prove (b).

Suppose that f is holomorphic. We introduce a holomorphic function $g \in \operatorname{Hol}(\Delta, \mathbb{C})$ as follows:

$$(21) \quad g(\lambda) := \langle f(\lambda\tau), \tau \rangle - (\beta/2)(\lambda^2 - 1).$$

It follows from (III) that

$$\operatorname{Re} \langle f(\lambda\tau), (\lambda\tau)^* \rangle \geq \beta/2.$$

On the other hand,

$$\begin{aligned} \langle f(\lambda\tau), (\lambda\tau)^* \rangle &= \left\langle f(\lambda\tau), \frac{\lambda\tau}{1-|\lambda|^2} - \frac{\tau}{1-\bar{\lambda}} \right\rangle = \langle f(\lambda\tau), \tau \rangle \cdot \left(\frac{\bar{\lambda}}{1-|\lambda|^2} - \frac{1}{1-\lambda} \right) \\ &= \left(g(\lambda) + \frac{\beta}{2}(\lambda^2 - 1) \right) \cdot \frac{\bar{\lambda} - 1}{(1-|\lambda|^2)(1-\lambda)} \\ &= g(\lambda) \cdot \frac{|\lambda - 1|^2}{-(1-|\lambda|^2)(1-\lambda)^2} + \frac{\beta(1-\bar{\lambda})(1+\lambda)}{2(1-|\lambda|^2)}. \end{aligned}$$

Therefore $\operatorname{Re} \frac{g(\lambda)}{-(1-\lambda)^2} \geq 0$. By the Riesz-Herglotz representation formula,

$$\frac{g(\lambda)}{-(1-\lambda)^2} = \oint_{|\zeta|=1} \frac{1 + \lambda\bar{\zeta}}{1 - \lambda\zeta} d\sigma(\zeta),$$

where $d\sigma(\zeta)$ is a positive measure on the unit circle. Decomposing $d\sigma$ with respect to Dirac's δ -function at the point $\zeta = 1$, $d\sigma(\zeta) = a\delta(\zeta) + d\sigma_1(\zeta)$, $a \geq 0$, we calculate:

$$(22) \quad \lim_{r \rightarrow 1^-} \frac{g(r)}{r-1} = \lim_{r \rightarrow 1^-} (1-r) \oint_{|\zeta|=1} \frac{1+r\bar{\zeta}}{1-r\zeta} (a\delta(\zeta) + d\sigma_1(\zeta)) = 2a \geq 0.$$

This fact, in turn, implies that

$$(23) \quad \lim_{r \rightarrow 1^-} \frac{\langle f(r\tau), \tau \rangle}{r-1} = \lim_{r \rightarrow 1^-} \left(\frac{g(r)}{r-1} + \frac{\beta}{2}(r+1) \right) = 2a + \beta$$

exists and is real.

Since by Step 4, $\operatorname{Re}(2a + \beta) = \beta$, we have that $a = 0$, and hence the limit

$$\lim_{r \rightarrow 1^-} \frac{\langle f(r\tau), \tau \rangle}{r-1} = \beta$$

is real. This completes the proof of our theorem. ■

Remark 2: As pointed out by the referee, our assumption that τ is a boundary null point of f in the sense that $\lim_{r \rightarrow 1^-} f(r\tau) = 0$ is used to prove the implication (I) \Rightarrow (III). As a matter of fact, our proof shows that (III) and (IV) are equivalent even without this assumption. For holomorphic mappings in the finite dimensional case, a complete characterization of boundary regular fixed points of semigroups is given in [6] (see Theorems 3.8 and 4.7 therein).

Example 2: Let \mathcal{H}_1 be a Hilbert space. Consider the semigroup of holomorphic self-mappings of the unit ball \mathbb{B} of the space $\mathcal{H} := \mathbb{C} \times \mathcal{H}_1$ defined by the formula

$$F_t(z_1, z_2) = \left(\frac{z_1}{z_1 + e^t(1 - z_1)}, \frac{e^{t/2}z_2}{z_1 + e^t(1 - z_1)} \right),$$

where $z_1 \in \mathbb{C}$, $z_2 \in \mathcal{H}_1$, $|z_1|^2 + \|z_2\|^2 < 1$. This semigroup has a boundary fixed point $\tau = (1, 0)$ which is not its Denjoy–Wolff point. Consider the semigroup generator $f \in \text{Hol}(\mathbb{B}, \mathcal{H})$:

$$f(z_1, z_2) = - \left. \frac{\partial F_t(z_1, z_2)}{\partial t} \right|_{t=0^+} = (z_1(1 - z_1), ((1 - 2z_1)z_2)/2).$$

Now we just calculate

$$\alpha = \lim_{r \rightarrow 1^-} \frac{\langle f(r\tau), \tau \rangle}{r - 1} = -1.$$

Hence,

$$d_\tau(F_t(z)) \leq e^t d_\tau(z), \quad z \in \mathbb{B}, t \geq 0.$$

Example 3: Define another semigroup $\{F_t\}_{t \geq 0} = \{(F_t)_1, (F_t)_2\}_{t \geq 0}$ on the unit ball \mathbb{B} of the same space \mathcal{H} by the formulas:

$$(F_t)_1(z_1, z_2) = \frac{(1 + z_1^2)e^{2t} - (1 - z_1)\sqrt{2(1 + z_1^2)e^{2t} - (1 - z_1)^2}}{(1 + z_1^2)e^{2t} - (1 - z_1)^2},$$

$$(F_t)_2(z_1, z_2) = z_2 \sqrt{\frac{\partial (F_t)_1(z_1, z_2)}{\partial z_1}},$$

where $z_1 \in \mathbb{C}$, $z_2 \in \mathcal{H}_1$, $|z_1|^2 + \|z_2\|^2 < 1$. Consider its generator $f \in \text{Hol}(\mathbb{B}, \mathcal{H})$:

$$f(z_1, z_2) = - \left. \frac{\partial F_t(z_1, z_2)}{\partial t} \right|_{t=0^+} = \left(-\frac{(1 - z_1)(1 + z_1^2)}{1 + z_1}, \frac{z_2(1 - z_1 + z_1^2 + z_1^3)}{(1 + z_1)^2} \right).$$

It is clear that f has three boundary null points: $\tau_1 = (1, 0)$, $\tau_2 = (i, 0)$ and $\tau_3 = (-i, 0)$. For each one of them we just calculate

$$\alpha_1 = \lim_{r \rightarrow 1^-} \frac{\langle f(r\tau_1), \tau_1 \rangle}{r - 1} = \lim_{r \rightarrow 1^-} \frac{-(1 - r)(1 + r^2)}{(1 + r)(r - 1)} = 1,$$

$$\alpha_2 = \lim_{r \rightarrow 1^-} \frac{\langle f(r\tau_2), \tau_2 \rangle}{r - 1} = \lim_{r \rightarrow 1^-} \frac{-(1 - ri)(1 - r^2)(-i)}{(1 + ri)(r - 1)} = -2,$$

$$\alpha_3 = \lim_{r \rightarrow 1^-} \frac{\langle f(r\tau_3), \tau_3 \rangle}{r - 1} = \lim_{r \rightarrow 1^-} \frac{-(1 + ri)(1 - r^2)i}{(1 - ri)(r - 1)} = -2.$$

Hence, the following three inequalities hold simultaneously for all $z \in \mathbb{B}$ and $t \geq 0$:

$$\begin{aligned} d_{\tau_1}(F_t(z)) &\leq e^{-t}d_{\tau_1}(z), \\ d_{\tau_2}(F_t(z)) &\leq e^{2t}d_{\tau_2}(z), \\ d_{\tau_3}(F_t(z)) &\leq e^{2t}d_{\tau_3}(z). \end{aligned}$$

These inequalities mean that for each point $z \in \mathbb{B}$ and for each $t \geq 0$, the image $F_t(z)$ belongs to the intersection of the ellipsoids:

$$E(\tau_1, e^{-t}d_{\tau_1}(z)) \cap E(\tau_2, e^{2t}d_{\tau_2}(z)) \cap E(\tau_3, e^{2t}d_{\tau_3}(z)).$$

Example 4: Consider the one-parameter continuous semigroup $\mathcal{S} = \{F_t\}_{t \geq 0}$, consisting of holomorphic self-mappings of the open unit disk Δ in the complex plane, defined by

$$F_t(z) = 1 - (1 - \exp(-t) + \exp(-t)\sqrt{1-z})^2, \quad z \in \Delta, \quad t \geq 0.$$

One can check that \mathcal{S} is generated by the following function:

$$f(z) = -2\sqrt{1-z}(\sqrt{1-z} - 1).$$

It is easy to see that $f(1) = 0$, but f has no angular derivative at the point $z = 1$. At the same time, this point is not even a fixed point of \mathcal{S} .

Example 5: Now we consider the semigroup $\mathcal{S} = \{F_t\}_{t \geq 0} \subset \text{Hol}(\Delta)$ defined by

$$F_t(z) = \frac{(1+z)^{\alpha(t)} - (1-z)^{\alpha(t)}}{(1+z)^{\alpha(t)} + (1-z)^{\alpha(t)}},$$

where $\alpha(t) = e^{-2t}$. Differentiating at $t = 0$, we find its generator:

$$f(z) = (1-z^2) \log \frac{1+z}{1-z}.$$

Similarly as in the previous example, $f(\pm 1) = 0$, but f has no angular derivative at the points $\tau_1 = 1$ and $\tau_2 = -1$. However, in contrast with that example, these points are fixed points of the semigroup:

$$F_t(\pm 1) = \pm 1 \quad \text{for all } t \geq 0.$$

Moreover, it is possible to calculate the “non-Euclidean distance” $d_{\pm 1}(F_t(z))$. In particular, for real $z = x$ we have:

$$d_1(F_t(x)) = \frac{1 - F_t(x)}{1 + F_t(x)} = ((1-x)/(1+x))^{\alpha(t)} = d_1(x)^{\alpha(t)}.$$

Since $0 < \alpha(t) < 1$ when $t > 0$, we conclude that for each fixed $t > 0$, the quotient $\frac{d_1(F_t(x))}{d_1(x)}$ tends to infinity as $z = x$ tends to 1 radially, i.e., $d_1(F_t(z))$ does not admit an estimate of the form $A(t)d_1(z)$.

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References

- [1] M. Abate, *Iteration Theory of Holomorphic Maps on Taut Manifolds*, Mediterranean Press, Rende, 1989.
- [2] M. Abate, *Angular derivatives in several complex variables*, in *Real Methods in Complex and CR Geometry*, Lecture Notes in Mathematics, 1848, Springer, Berlin, 2004, pp. 1–47.
- [3] M. Abate and R. Tauraso, *The Julia–Wolff–Carathéodory theorem(s)*, in *Complex geometric analysis in Pohang (1997)*, Contemporary Mathematics, 222, 1999, pp. 161–172.
- [4] D. Aharonov, M. Elin, S. Reich and D. Shoikhet, *Parametric representations of semi-complete vector fields on the unit balls in \mathbb{C}^n and in Hilbert space*, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Serie IX. Matematica e Applicazioni **10** (1999), 229–253.
- [5] L. Aizenberg and D. Shoikhet, *Boundary behavior of semigroups of holomorphic mappings on the unit ball in \mathbb{C}^n* , Complex Variables. Theory and Application. An International Journal **47** (2002), 109–121.
- [6] F. Bracci, M. D. Contreras and S. Díaz-Madrigal, *Pluripotential theory, semigroups and boundary behavior of infinitesimal generators in strongly convex domains*, preprint, arXiv.math.CV/0607670, 2006.
- [7] M. D. Contreras, S. Díaz-Madrigal and Ch. Pommerenke, *On boundary critical points for semigroups of analytic functions*, Mathematica Scandinavia **98** (2006), 125–142.
- [8] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, FL, 1995.
- [9] S. Dineen, *The Schwarz Lemma*, Clarendon Press, Oxford, 1989.
- [10] M. Elin, S. Reich and D. Shoikhet, *Asymptotic behavior of semigroups of ρ -nonexpansive and holomorphic mappings on the Hilbert ball*, Annali di Matematica Pura ed Applicata. Series IV **181** (2002), 501–526.
- [11] M. Elin and D. Shoikhet, *Dynamic extension of the Julia–Wolff–Carathéodory theorem*, Dynamic Systems and Applications **10** (2001), 421–437.
- [12] M. Elin and D. Shoikhet, *Semigroups of holomorphic mappings with boundary fixed points and spirallike mappings*, in *Geometric Function Theory in Several Complex Variables*, World Sci. Publ., River Edge, NJ, 2004, pp. 82–117.
- [13] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.
- [14] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

- [15] L. A. Harris, S. Reich and D. Shoikhet, *Dissipative holomorphic functions, Bloch radii, and the Schwarz lemma*, Journal d'Analyse Mathématique **82** (2000), 221–232.
- [16] E. Kopecká and S. Reich, *Hyperbolic monotonicity in the Hilbert ball*, *Fixed Point Theory and Applications*, 2006, Article ID 78104, pp. 1–15.
- [17] R. H. Martin, Jr., *Differential equations on closed subsets of a Banach space*, Transactions of the American Mathematical Society **179** (1973), 399–414.
- [18] Ch. Pommerenke, *Boundary Behavior of Conformal Maps*, Springer, Berlin, 1992.
- [19] S. Reich, *On fixed point theorems obtained from existence theorems for differential equations*, Journal of Mathematical Analysis and Applications **54** (1976), 26–36.
- [20] S. Reich and D. Shoikhet, *Generation theory for semigroups of holomorphic mappings in Banach spaces*, Abstract and Applied Analysis **1** (1996), 1–44.
- [21] S. Reich and D. Shoikhet, *Semigroups and generators on convex domains with the hyperbolic metric*, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Serie IX. Matematica e Applicazioni **8** (1997), pp. 231–250.
- [22] S. Reich and D. Shoikhet, The Denjoy–Wolff theorem, *Encyclopaedia of Mathematics*, Supplement III, Kluwer Academic Publishers, Dordrecht, 2001, pp. 121–123.
- [23] S. Reich and D. Shoikhet, *Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces*, Imperial College Press, London, 2005.
- [24] W. Rudin, *Function Theory in the Unit Ball of \mathbb{C}^n* , Springer, Berlin, 1980.
- [25] J. H. Shapiro, *Composition Operators and Classical Function Theory*, Springer, Berlin, 1993.
- [26] D. Shoikhet, *Semigroups in Geometrical Function Theory*, Kluwer, Dordrecht, 2001.
- [27] K. Yosida, *Functional Analysis*, Springer, Berlin, 1980.